

Who has seen a free photon?

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Abstract

While the notion of the position of photons is indispensable in the quantum optical situations, it has been known in mathematical physics that any position operator cannot be defined for a massless free particle with a non-zero finite spin. This dilemma is resolved by introducing the “effective mass” of a photon due to the interaction with matter. The validity of this interpretation is confirmed in reference to the picture of “polariton”, a basic notion in optical and solid physics. In this connection, we discuss the long-standing controversy between Minkowski’s and Abraham’s definitions of the momenta of a photon in media from the general viewpoint adopted in the appendix.

1 A dilemma about localizability of photons

In the recent advanced quantum-optical technology, the notion of positions of photons has played indispensable roles in the experimental situations, as is exemplified by the most familiar Mach-Zender interferometry consisting of the beam splitters and of the photon detector to specify the position of detection of photons. Theoretically speaking, this is the contexts where the localization of a photon is to be described by means of some *position observables* associated with the quantum electromagnetic field. Here we encounter the following serious difficulty: since the paper of Newton and Wigner [12], it has been known in mathematical physics that any position operator cannot be defined for a massless free particle with a non-zero finite spin, in sharp contrast to the cases of massive particles which can be localized. This statement is clearly in contradiction to the above familiar situations where almost all physicists have used the notion of “position of a photon” as one of the basic ingredients of theory and application of quantum mechanics. Then, who has seen a free photon?

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Once Dirac wrote in [4] as follows:

If we are given a beam of roughly monochromatic light, then we know something about the location and momentum of the associated photons. We know that each of them is located somewhere in the region of space through which the beam is passing
...

This modest statement contradicts, however, the above general result, if it is literally interpreted. Namely, if we try to make sense out of the sentence, “each free photon is located somewhere in the region of space”, we need to introduce a “position operator for a free photon in three-dimensional space as a well defined observable”, which is in contradiction with [12]. In the present paper, we propose a solution to the problem of localization of a photon taking proper account of the relevant physical interactions between a photon to be localized and the matter to localize the former, combining the basic ideas relevant to both fields.

2 Strategy to solve the dilemma

We propose the following resolution: while a massless free photon is *not* localizable, a “real” photon can be made localizable by its dynamical interaction with matter such as media or devices (for its detection). To be precise, this simple picture can be shown to have a sound basis as follows. First we see in Section 3 that a photon can be localized only if photon-matter interaction provides a positive effective mass, by our reinterpretation of Wightman’s theorem [17] following from the arguments by Newton and Wigner [12]. Then we show in Section 4 how the photon-matter interaction can be reduced in this context into the picture of “a free particle with positive effective mass”. This interpretation becomes clearer if we refer to the picture of “polariton” [7, 6], a basic notion in optical and solid-state physics.

This kind of new connection between mathematical physics and other region of physics should bring fruitful perspectives. We conclude this paper with a re-interpretation of Dirac’s statement above and further suggestions.

3 Formulation of the problem

3.1 Newton-Wigner-Wightman analysis

In 1949, Newton and Wigner [12] raised the question of localizability of single free particles. They attempted to formulate the properties of the localized states on the basis of natural requirements of relativistic covariance.

Physical quantities available in this formulation admitting direct physical meaning are restricted inevitably to the generators of Poincaré group $\mathcal{P}_+^\uparrow = \mathbb{R}^4 \rtimes L_+^\uparrow$ (with L_+^\uparrow the orthochronous proper Lorentz group) which is locally isomorphic to the semi-direct product $\mathcal{H}_2(\mathbb{C}) \rtimes SL(2, \mathbb{C})$ of the hermitian (2×2) -matrices and of $SL(2, \mathbb{C})$, consisting of the energy-momentum vector P_μ and of the Lorentz generators $M_{\mu\nu}$ (composed of angular momenta M_{ij} and of Lorentz boosts M_{0i}). The problem is then to find conditions under which “position operators” can naturally be constructed from the Poincaré generators $(P_\mu, M_{\mu\nu})$. In [12], position operators have been shown to exist in massive cases in an essentially unique way for “elementary” systems in the sense of the irreducibility of the corresponding representations of \mathcal{P}_+^\uparrow so that localizability of a state can be defined in terms of such position operators. In massless cases, however, no localized states are found to exist in the above sense. That was the beginning of the story.

Wightman [17] clarified the situation by recapturing the notion of “localization” in a general form as follows. First he has reformulated the usual approaches in terms of unbounded position operators into the form of general axioms (i)-(v) involving projection operators,

- (i) To each Borel subset Δ of \mathbb{R}^3 , there corresponds a projection operator $E(\Delta)$ in a Hilbert space \mathfrak{H} , whose expectation value gives the probability of finding the system in Δ ;
- (ii) $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$;
- (iii) $E(\Delta_1 \cup \Delta_2) = E(\Delta_1) + E(\Delta_2)$, if $\Delta_1 \cap \Delta_2 = \emptyset$;
- (iv) $E(\mathbb{R}^3) = 1$;
- (v) $E(\mathcal{R}\Delta + \mathbf{a}) = U(\mathbf{a}, \mathcal{R})E(\Delta)U(\mathbf{a}, \mathcal{R})^{-1}$, where $\mathcal{R}\Delta + \mathbf{a}$ is the set obtained from Δ by applying a translation \mathbf{a} after a rotation \mathcal{R} , and $U(\mathbf{a}, \mathcal{R})$ is the corresponding unitary operator in \mathfrak{H} .

Note that the notion of localizability discussed above is concerned with *localization of states in space at a given time*. If we consider the axioms like (i)-(v) on the whole space-time, it would imply the validity of the CCR relations between 4-momenta p_μ and space-time coordinates x^ν , which would imply the Lebesgue spectrum covering the whole \mathbb{R}^4 for both observables \hat{p}_μ and \hat{x}^ν . Then, any such physical requirements as the spectrum condition cannot be imposed on the energy-momentum spectrum \hat{p}_μ , and hence, the notion of localizability in space-time does not make sense.

According to Mackey’s theory of induced representations, Wightman’s formulation can easily be seen as the condition for the set of operators $\{E(\Delta)\}$ to constitute a *system of imprimitivity* [9] under the action of the unitary representation $U(a, \mathcal{R})$ in \mathfrak{H} of the three-dimensional Euclidean

group $SE(3) := \mathbb{R}^3 \rtimes SO(3)$. In a more algebraic form, the pair (E, U) can also be viewed as a *covariant representation*

$$E(\tau_{(\mathbf{a}, \mathcal{R})}(f)) = U(a, \mathcal{R})E(f)U(a, \mathcal{R})^{-1} \quad \text{for } f \in L^\infty(\mathbb{R}^3), (\mathbf{a}, \mathcal{R}) \in SE(3), \quad (1)$$

of an action $SE(3) \curvearrowright_\tau L^\infty(\mathbb{R}^3)$, $[\tau_{(\mathbf{a}, \mathcal{R})}(f)](\mathbf{x}) := f(\mathcal{R}^{-1}(\mathbf{x} - \mathbf{a}))$ on the algebra $L^\infty(\mathbb{R}^3)$ generated by the position operators in the representation $E : L^\infty(\mathbb{R}^3) \ni f \mapsto E(f) = \int f(\mathbf{x})dE(\mathbf{x}) \in B(\mathfrak{H})$, s.t. $E(\chi_\Delta) = E(\Delta)$.

Thus Wightman's formulation of the Newton-Wigner localizability problem is just to examine whether the Hilbert space \mathfrak{H} of the representation (U, \mathfrak{H}) of $SE(3)$ can accommodate a representation E of the algebra $L^\infty(\mathbb{R}^3)$ consisting of position operators, covariant under the action of $SE(3)$ in the sense of (1).

Applying the general theory of Mackey to the case of three-dimensional Euclidean group $SE(3)$, Wightman proved the fundamental result below as a purely kinematical consequence.

Theorem 1 ([17], excerpt from theorem 6 and 7) *A Lorentz or Galilei covariant massive system is always localizable. For the Lorentz case, the only localizable massless elementary system (i.e. irreducible representation) has spin zero. For the Galilei case, no massless elementary system is localizable.*

Corollary 2 *A free photon is not localizable.*

The essential mechanism of (non-)localizability in the sense of Newton-Wigner-Wightman depends on the structure of little groups defined by Wigner as the stabilizer groups of standard four-momenta on each type of \mathcal{P}_+^\uparrow -orbits in p -space.

On an orbit $p^2 = m^2c^2 > 0$ under \mathcal{P}_+^\uparrow , we can choose a standard momentum $p^{(0)} := (mc, \mathbf{0})$ which specifies a rest frame of a particle with mass $m \neq 0$. Then, the little group at $p^{(0)}$ is the group $SO(3)$ of spatial rotations, corresponding to the degrees of freedom remaining in the rest frame. As a consequence, “the space of all Lorentz frames” along the orbit becomes $SO(1, 3)/SO(3) \cong \mathbb{R}^3$. Note that a Lorentz boost Λ_p defined by

$$\Lambda_p = \begin{pmatrix} \frac{p^0}{mc} & \frac{t\mathbf{p}}{mc} \\ \frac{\mathbf{p}}{mc} & \mathbf{1} + (\frac{p^0}{mc} - 1)\frac{\mathbf{p}^t\mathbf{p}}{p^2} \end{pmatrix} \in SO(1, 3)$$

transforms $p^{(0)}$ into $p = \Lambda_p p^{(0)} = (p^0, \mathbf{p}) = (\sqrt{m^2c^2 + \mathbf{p}^2}, \mathbf{p})$. On the other hand, we have for $p_\Lambda = \Lambda_p p^{(0)}$ a relative velocity $\mathbf{u}_\Lambda \in \mathbb{R}^3 \cong SO(1, 3)/SO(3)$

between the Lorentz frames $(1, \mathbf{0})$ and $\frac{p_\Lambda}{mc} =: u_\Lambda = (\frac{1}{\sqrt{1 - \mathbf{u}_\Lambda^2/c^2}}, \frac{\mathbf{u}_\Lambda/c}{\sqrt{1 - \mathbf{u}_\Lambda^2/c^2}})$,

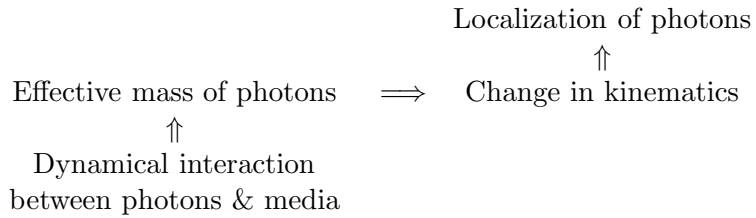
$u_\Lambda^2 = 1$. Thus, the homeomorphism $\mathbb{R}^3 \cong SO(1,3)/SO(3)$ describes a non-trivial action of $\mathbf{u} = \mathbf{u}_\Lambda \in \mathbb{R}^3$ on $p = (p^0, \mathbf{p})$ belonging to the \mathcal{P}_+^\uparrow -orbit $p^2 = m^2 c^2 > 0$ through the action of Lorentz boost $\Lambda = \Lambda(\mathbf{u}) = \Lambda_p \in SO(1,3)$ transforming $p^{(0)}$ into $p = \Lambda p^{(0)}$. Hence the coordinates $\mathbf{u} \in \mathbb{R}^3 \cong SO(1,3)/SO(3)$ of Lorentz frames just play the role of the order parameters (or, “sector parameters”) on each \mathcal{P}_+^\uparrow -orbit as the space of “condensation” associated with a symmetry breaking of boost invariance, and hence, \mathbf{u} can be identified with position operators in the imprimitivity system appearing in Wightman’s theorem.

In sharp contrast, there is no rest frames for a massless particle and the little group becomes isomorphic to two-dimensional Euclidean group $SE(2)$, whose rotational generator corresponds to the helicity. Since the other two translation generators corresponding to gauge transformations span *non-compact* directions in distinction from the massive cases with compact $SO(3)$, the allowed representation is only the trivial one which leaves the transverse modes invariant, and hence, the little group cannot provide position operators in the massless case.

Since Newton-Wigner-Wightman, many discussions around the photon localization problem have been developed. So far as we know, the arguments seem to be divided into two opposite viewpoints, one relying on purely dynamical bases [5] and another on pure kinematics [2], where it is almost impossible to find any meaningful agreements. Below we propose an alternative strategy based on the notion of “effective mass”, which can provide a reasonable reconciliation between these conflicting ideas because of its “kinematical” nature arising from some dynamical origin.

3.2 Wightman’s theorem as the “basis” for localization

Our scheme of the localization for photons can be summarized as follows, which is essentially in accordance with the basic formulation of “quadrality scheme” [14] underlying the Micro-Macro duality [13]:



Once a positive effective mass appears, Wightman’s theorem itself provides the “kinematical basis” for the localization of a photon. From our point of view, therefore, this theorem so far regarded as a no-go theorem against the localizability becomes actually an affirmative support for it, conveying such a strongly selective meaning that, whenever a photon is localized, it should carry a non-zero effective mass.

In the next section, we explain the meaning of our scheme from a physical point of view.

4 Resolution of the dilemma

4.1 How to define effective mass of a photon

Now we focus on a photon interacting with homogeneous medium, in the case of the monochromatic light with angular frequency ω as a classical light wave. For simplicity, we neglect here the effect of absorption, that is, the imaginary part of refractive index. When a photon interacting with matter can be treated as a single particle, it is natural to identify its velocity \mathbf{v} with the “signal velocity” of light in medium. The relativistic total energy of the particle E should be related to $v := \sqrt{\mathbf{v} \cdot \mathbf{v}}$ by its mass m_{eff} :

$$E = \frac{m_{\text{eff}} c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (2)$$

Since v is well known to be smaller than the light velocity c (theoretically or experimentally), m_{eff} is positive (when the particle picture above is valid). Then we may consider m_{eff} as the relativistic “effective (rest) mass of a photon”, and identify its momentum \mathbf{p} with

$$\mathbf{p} = \frac{m_{\text{eff}} \mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (3)$$

Hence, as long as “an interacting photon” can be approximately treated as a single particle, it should be massive, according to which its “localization problem” is resolved. The validity of this picture will be confirmed in the next subsection.

The concrete forms of energy/momentum are related to the Abraham-Minkowski controversy [1, 11, 3] and modified versions of Einstein/de Broglie formulae. (We discuss this point in the appendix.)

Our argument itself, however, does not depend on the energy/momentum formulae. The only essential point is that the interactoin can make a massive particle from a massless one. That is, while a free photon satisfies

$$E_{\text{free}}^2 - c^2 p_{\text{free}}^2 = 0, \quad (4)$$

an interacting photon satisfies

$$E^2 - c^2 p^2 = m_{\text{eff}}^2 c^4 > 0. \quad (5)$$

To sum up, an “interacting photon” can gain a positive effective mass, while a “free photon” remains massless! This is the key we have sought for. However, the argument in this section is based on the assumption that “a photon dressed with interaction” can be viewed as a single particle. Then we

proceed to consider the validity of our picture, especially the existence of particles whose effective mass is obtained by the interaction, analogous to Higgs mechanism: Such a universal model for photon localization exists. It is the notion of polariton, well known in optical and solid physics.

4.2 Polaritons as a universal model for photon localization

In optical and solid-state physics, the propagation of light in a medium is viewed as follows: By the interaction between light and matter, creation of an “exciton (an excited state of polarization field above the Fermi surface)” and annihilation of a photon will be followed by annihilation of an exciton and creation of a photon, ..., and so on. This chain of processes itself is often considered as the motion of particles called polaritons (in this case “exciton-polaritons”), which constitute particles associated with the coupled wave of the polarization wave and electromagnetic wave.

Remark 3 *In spite of the similarity in its name, a polariton should not be confused with a polaron [8] which represents a fermion as a charged matter dressed by polarization field. In contrast, a polariton is a boson which represents a “dressed photon”.*

The notion of polariton has been introduced to develop the microscopic theory of electromagnetic interactions in materials ([7], [6]). An injected photons become polaritons by the interaction with matter. As exciton-phonon interaction is dissipative, the polariton picture gives a scenario of absorption. It has provided a better approximation than the scenarios without a polariton. Moreover, the group velocity of polaritons discussed below gives another confirmation of the presence of an effective mass.

As is well known, permittivity $\epsilon(\omega)$ is given by the following equality,

$$\epsilon(\omega) = n^2 = \frac{c^2 k^2}{\omega^2}, \quad (6)$$

and hence, we obtain the dispersion relation (a relation between frequency and wave number) of polariton once the formula of permittivity is given.

Remark 4 *In general, this dispersion relation implies branching, analogous to the Higgs mechanism. The signal pulse corresponding to each branches can also be detected in many experiments, for example, in [10] cited below.*

In the simple case, the permittivity is given by the transverse frequency ω_T of exciton’s (lattice vibration) as follows:

$$\epsilon(\omega) = \epsilon_\infty + \frac{\omega_T^2(\epsilon_{st} - \epsilon_\infty)}{\omega_T^2 - \omega^2}, \quad (7)$$

where ϵ_∞ denotes $\lim_{\omega \rightarrow \infty} \epsilon(\omega)$ and $\epsilon_{st} = \epsilon(0)$ (static permittivity). With a slight improvement through the wavenumber dependence of the exciton energy, the theoretical result of polariton group velocity $\frac{\partial \omega}{\partial \mathbf{k}} < c$ based on the above dispersion relation can explain satisfactorily experimental data of the passing time of light in materials (for example, [10]). This strongly supports the validity of the polariton picture.

From the above arguments, polaritons can be considered as a universal model of the “interacting photons in a medium” in the previous subsection 4.1. The positive mass of a polariton gives a solution to its “localization problem”. Conversely, as the “consequence” of Wightman’s theorem, it follows that “all” physically accessible photons as particles which can be localized are more or less polaritons (or similar particles) because only the interaction can give a photon its effective mass, if it does not violate particle picture. In this way, the dilemma between Newton-Wigner-Wightman theorem and the position observable of photons is successfully resolved by combining useful mathematical methods and meaningful physical concepts, which were separated before causing a negative result.

5 Concluding remarks

Now the statement by Dirac in the beginning section can be justified in the following form: If we are given a beam of roughly monochromatic light, then we know something about the location and momentum of the associated photons with “effective mass” (polaritons) arising from their interactions with matter media. We know that each of them is located somewhere in the region of space “filled with a medium (like the air or crystals)” through which the beam is passing ...

This modified statement and all the discussions in the present paper clarify the important roles played by interactions in making sense of the notion of localization. We can expect that the discussion in the present paper will shed some new light on the idea of the “emergence” of space-time proposed in [14]. In combination with a possible scenario for the mass generation, we can summarize the argument above in the following quadrality scheme:

$$\begin{array}{ccc}
 & & x : \begin{array}{l} \text{localization} \\ \text{of photons} \end{array} \\
 & & \uparrow \\
 m_{\text{eff}} : \text{Effective mass of photon} & \implies & v : \text{kinematics} \\
 \uparrow & & \\
 p : \begin{array}{l} \text{dynamical interaction} \\ \text{between photons \& media} \end{array} & &
 \end{array}$$

When all the above ingredients are established, the “mechanics of mass points” becomes meaningful. A photon, which is something quite dissimilar

to a “mass point”, appears ubiquitously since the electromagnetic field works as a universal medium to mediate the interaction between charged particles which provides an idealized standard reference system. The answer to our question at the beginning can now be found in the following modified form of the famous verse [16]:

Who has seen a free photon?
Neither I nor you.
But when the matter reacts trembling
the photons are passing through.

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Appendix: A new viewpoint on Minkowski-Abraham controversy and generalized Einstein-de Broglie formula

We consider in this appendix the problem as to how to express the effective mass m_{eff} of a photon (a polariton) in a medium precisely, which can be easily answered if E or \mathbf{p} is identified. At this point, however, we encounter another mystery which has been the origin of a long-standing controversy for one century(!), known as “Abraham-Minkowski dilemma” [3] concerning the correct formula for the momentum of a photon in medium: in 1908 Minkowski [11] proposed a candidate for the energy momentum tensor of the electromagnetic field in a medium, according to which the photon momentum takes such a form as

$$p = p_{\text{free}} \cdot n, \quad (8)$$

where $p := \sqrt{\mathbf{p} \cdot \mathbf{p}}$ is the magnitude of the momentum and p_{free} the free photon momentum. n denotes the refractive index and v_{ph} is the magnitude of phase velocity of light in the medium, as is given as usual by

$$v_{\text{ph}} = \frac{\omega}{k} = \frac{c}{n}, \quad (9)$$

where k and ω are, respectively, the magnitude of the wavenumber \mathbf{k} and the frequency of the (classical) light wave in the medium. When $v_{\text{ph}} < c$ or

equivalently $n > 1$ (as it should be in the normal situations), $p = p_{Min}$ is larger than the free photon momentum p_{free} . On the other hand, Abraham [1] proposed in 1909 another version leading to a formula for the momentum $p = p_{Abr}$ of a photon in a medium given by

$$p = p_{free} \cdot \frac{v}{c}, \quad (10)$$

which is smaller than pure photon momentum.

To settle the matter, we start from the Minkowski-type momentum for a photon in a medium:

$$\mathbf{p} = \hbar \mathbf{k}. \quad (11)$$

with

$$p = p_{free} \cdot n = p_{free} \cdot \frac{c}{v_{ph}} = \frac{\hbar \omega}{v_{ph}}. \quad (12)$$

Then the effective mass m_{eff} is given by

$$m_{eff} = \frac{\sqrt{1 - \frac{v^2}{c^2}}}{v_{ph}v} \hbar \omega \quad (13)$$

and hence, the energy is

$$E = \frac{c^2}{v_{ph}v} \hbar \omega. \quad (14)$$

The above formula for the energy can be considered as a generalization of Einstein-Planck formula for “a photon in a medium”. The factor

$$\frac{c^2}{v_{ph}v} = \frac{nc}{v} \quad (15)$$

is due to the interaction with matter and becomes 1 in the case of free photons or other free particles.

Remark 5 *In terms of a modified angular frequency $\tilde{\omega} = \frac{c^2}{v_{ph}v} \omega$, this energy can be represented as $E = \hbar \tilde{\omega}$, just similarly to the matter wave frequency for a free massive particle. It is, however, sufficient for us to make use of ω (classical wave frequency) in the present paper.*

On the other hand, if we adopt the Abraham type formula

$$p = p_{free} \cdot \frac{v}{c} = \frac{\hbar \omega v}{c^2} = \frac{v_{ph}v}{c^2} \hbar k \quad (16)$$

then m_{eff} and E should be given by

$$m_{eff} = \frac{\sqrt{1 - \frac{v^2}{c^2}}}{c^2} \hbar \omega \quad (17)$$

and

$$E = \hbar\omega. \quad (18)$$

This is identical to usual Einstein-Planck formula, where the Abraham momentum itself violates the usual de Broglie formula.

To sum up, Minkowski picture provides

$$\begin{aligned} \mathbf{p} &= \hbar\mathbf{k}. \\ E &= \frac{c^2}{v_{\text{ph}}v} \hbar\omega \end{aligned} \quad (19)$$

while Abraham picture provides

$$\begin{aligned} \mathbf{p} &= \frac{v_{\text{ph}}v}{c^2} \hbar\mathbf{k}. \\ E &= \hbar\omega \end{aligned} \quad (20)$$

As we have seen above, the precise form of the effective mass of a photon in a medium is in fact related to the question which generalized Einstein-de Broglie relation for the case with interaction is suitable. This also suggests the role of our effective mass as an order parameter due to certain kind of symmetry breaking.

Perhaps (in the case of $n > 1$) the Abraham type momentum, energy and mass can be considered as those of “pure photon components” in a polariton, whose momentum (sum of momenta for all branches) is showed to be Minkowski type [3]. Further discussion will be done in a succeeding paper [15].

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